Concerning the Twisted Biquadratic.

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Among the covariant figures of the twisted biquadratic R there exists a determinate tetrahedral quartic surface F, whose relation to the curve will be the subject of the following paper.

Let R be given as the intersection of two quadrics S and S', then, by a known theorem, the coordinates of a point on the curve are rational expressions of the elliptic functions, obtained by inverting the elliptic integral

$$u = \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\Delta'\xi^4 + \theta'\xi^3 + \phi\xi^2 + \theta\xi + \Delta}} = \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{G(\xi)}},$$

where Δ', θ', ϕ, θ, Δ denote in the usual manner the common invariants of S and S', and in this way to each point on the curve a determinate argument u becomes affixed.

If we make the lower limit ξ_0 of the integral a root of $G(\xi) = 0$, the doubly-periodic function $\xi(u)$ will be even, and therefore it will only be some linear transformation of Weierstrass's function $\wp u$. At the same time the derived function $\xi'(u)$ will be odd and $\xi'(u) = \pm\sqrt{G(\xi)}$ will have the four zeros $0, \omega_1, \omega_2, \omega_3$, where $\omega_1, \omega_2, \omega_3$ are three half-periods, satisfying the relation $\omega_1 + \omega_2 + \omega_3 = 0$.

Every plane now intersects R in four points, the arguments of which have a sum equal to zero, and consequently on a quadric through R every right line of the first system meets the curve in two points, whose arguments have the constant sum v, whereas for the right lines of the other system that sum is equal to $-v$. For the four cones, which pass through R, the two systems coincide, hence with these surfaces are associated the values $v = 0, \omega_1, \omega_2, \omega_3$.

We can ask for the value of v, belonging to the quadric $S + \mu S'$. Evidently to a given value of μ there are corresponding two values of v, only differing in sign, conversely v being given, only one value of μ can be found. From this we

infer that μ is some linear function of $\xi(v)$, but as we have for the four cones $G(\mu)=0$, and as the roots of the same equation are precisely the values which $\xi(v)$ assumes for $v=0$, ω_1, ω_2, ω_3, it is clear that the parameter μ of the quadric $S+\mu S'$ is simply equal to $\xi(v)$.

It is now possible to discriminate the quadrics of the system $S+\mu S'$ according to the different values of v, and in doing so, we are led to consider the surfaces corresponding to rational parts of the periods. More particularly, we may notice the six surfaces belonging to the quarter-periods, which, according to Ameseder,* were first studied by Voss.† We shall call, for shortness, these surfaces the Vossian quadrics of the curve R, and shall denote them by H_1, H_{I}, H_2, H_{II}, H_3, H_{III}, the six associated values of v being

$$\frac{\omega_1}{2}\,,\ \frac{\omega_1}{2}+\omega_2,\ \frac{\omega_2}{2}\,,\ \frac{\omega_2}{2}+\omega_3,\ \frac{\omega_3}{2}\,,\ \frac{\omega_3}{2}+\omega_1.$$

From these we get only three distinct values of $2v$, therefore the Vossian quadrics may be arranged in three pairs of conjugate surfaces: H_1 and H_{I}, H_2 and H_{II}, H_3 and H_{III}, each pair corresponding to one of the three half-periods. We next examine the values of the parameter μ for the Vossian quadrics. By the properties of the function $\wp u$ we know that if $\wp 0$, $\wp\omega_1$, $\wp\omega_2$, $\wp\omega_3$ are the roots of a binary quartic, the roots of its sextic covariant are the values of $\wp v$ for the quarter-periods. Now since $\xi(v)$ is only a linear transformation of $\wp v$, the same theorem holds good for $\xi(v)$, and so it is apparent that the required values of μ are the zeros of the sextic covariant $T(\mu)$, derived from the fundamental quartic $G(\mu)$.

Let us suppose that the two quadrics S and S' are a pair of conjugate Vossian quadrics, then 0 and ∞ form one of the three pairs of conjugate roots of $T(\mu)$, hence in the quartic $G(\mu)$ only even powers of μ occur, and we thus get the theorem: For each pair of conjugate Vossian quadrics the common invariants θ' and θ will vanish identically. Reciprocating this theorem, we can prove that as soon as the invariants θ' and θ are vanishing, the quadrics S and S' are a pair of conjugate Vossian quadrics with respect to their intersection.

It is easily shown, by means of the special values of v, that on any one of these Vossian quadrics we can place the sides of an infinity of skew quadrilate-

* "Ueber Configurationen auf der Raumcurve 4ter Ordnung, 1ster Species"; Wiener Sitzungsberichte, Bd. 87, p. 1194.

† "Die Liniengeometrie in ihrer Anwendung auf die Flächen 2ten Grades"; Math. Annalen, Bd. 10.

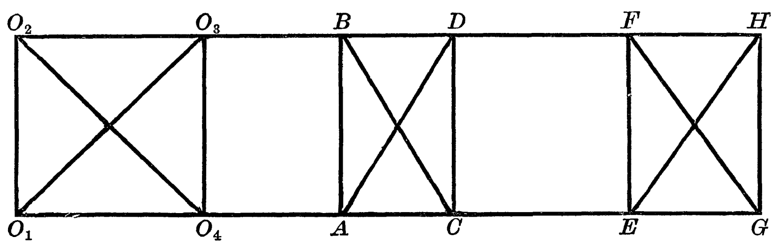
rals inscribed in the biquadratic R. Moreover we can find on each of these surfaces four chords XY of the curve R, such that three times the argument of either extremity added to the argument of the other one is equal to zero. Otherwise stated, we can find on a Vossian quadric four chords XY, each of which is the intersection of the osculating planes in its extremities. So for example on H_1, corresponding to $v = \frac{\omega_1}{2}$, we have only to join the points X and Y with the affixes

$$X \ldots \frac{\omega_1}{4} + \omega_2, \quad -\frac{3\omega_1}{4} + \omega_2, \quad -\frac{\omega_1}{4} + \omega_2, \quad -\frac{3\omega_1}{4} + \omega_2,$$

$$Y \ldots \frac{\omega_1}{4} + \omega_3, \quad -\frac{3\omega_1}{4} + \omega_3, \quad -\frac{\omega_1}{4} + \omega_3, \quad -\frac{3\omega_1}{4} + \omega_3,$$

to obtain four chords XY possessing this characteristic property.

In all we have 24 of these chords, which we shall call the *chords of curvature* of R. Their position with respect to R and with respect to the self-conjugate tetrahedron $O_1 O_2 O_3 O_4$, associated with that curve, is very symmetrical, but as it is amply discussed in the cited paper of Ameseder and in other papers dealing with the theory of the biquadratic, it may suffice to mention here that the chords of curvature can be divided into three groups of 8, all chords of one group meeting the same pair of opposite edges of the tetrahedron $O_1 O_2 O_3 O_4$. In the accompanying schematical diagram one of these groups is represented; through four of the eight chords AB, BC, CD, DA, forming the sides of a skew



quadrilateral $ABCD$ whose vertices lie upon the opposite edges $O_2 O_3$ and $O_1 O_4$, the Vossian quadric H_1 can be described, similarly through the other four EF, FG, GH, HE passes the conjugate quadric H_{I}. Further investigation proves that each of the three pairs of points O_1, O_4; A, C; E, G on the edge $O_1 O_4$, likewise each of the three pairs O_2, O_3; B, D; F, H on the edge $O_2 O_3$, may be regarded as the pair of foci of the involution defined by the two remaining pair of points. Thus, then, it is evident that the complete group of

eight chords can be constructed without ambiguity as soon as one of them, AB for instance, is placed at random on the edges O_2O_3 and O_1O_4 of the self-conjugate tetrahedron, which henceforth we shall use as the tetrahedron of reference. Let AB be represented by the equations $x=\alpha w$, $y=\alpha z$, where α denotes $e^{\frac{\pi i}{4}}$, then we can write down the equations of all the chords of the group as follows:

$$AB\begin{cases}x=\alpha w\\ y=\alpha z\end{cases},\quad BC\begin{cases}x=-\alpha w\\ y=\alpha z\end{cases},\quad CD\begin{cases}x=-\alpha w\\ y=-\alpha z\end{cases},\quad DA\begin{cases}x=\alpha w\\ y=-\alpha z\end{cases},$$
$$EF\begin{cases}x=\alpha^3 w\\ y=\alpha^3 z\end{cases},\quad FG\begin{cases}x=-\alpha^3 w\\ y=\alpha^3 z\end{cases},\quad GH\begin{cases}x=-\alpha^3 w\\ y=-\alpha^3 z\end{cases},\quad HA\begin{cases}x=\alpha^3 w\\ y=-\alpha^3 z\end{cases}.$$

But though the eight chords are now defined, they fail to determine completely the equations of the Vossian quadrics H_1 and H_{I} and of their intersection R. Introducing two arbitrary constants λ and μ, we only know that these equations can be thrown into the form

$$H_1=\lambda(x^2-iw^2)+(y^2-iz^2)=0,$$
$$H_{\mathrm{I}}=\mu(x^2+iw^2)+(y^2+iz^2)=0;$$

for each of these quadrics contains the four sides of one of the previously considered quadrilaterals, and, as an easy reckoning shows, the conditions $\theta'=0$, $\theta=0$, characteristic for a pair of conjugate Vossian quadrics, are fulfilled. Passing now from H_1 and H_{I} to the remaining pairs of Vossian quadrics, we get without much difficulty for their equations

$$\begin{cases}H_2=\sqrt{\mu}H_1+i\sqrt{\lambda}H_{\mathrm{I}}\\ \qquad =(\sqrt{\lambda}+i\sqrt{\mu})(x^2\sqrt{\lambda\mu}-z^2)+(\sqrt{\mu}+i\sqrt{\lambda})(y^2-w^2\sqrt{\mu\lambda})=0,\\ H_{\mathrm{II}}=\sqrt{\mu}H_1-i\sqrt{\lambda}H_{\mathrm{I}}\\ \qquad =(\sqrt{\lambda}-i\sqrt{\mu})(x^2\sqrt{\lambda\mu}+z^2)+(\sqrt{\mu}-i\sqrt{\lambda})(y^2+w^2\sqrt{\mu\lambda})=0,\end{cases}$$

$$\begin{cases}H_3=\sqrt{\mu}H_1+\sqrt{\lambda}H_{\mathrm{I}}\\ \qquad =(\sqrt{\lambda}+\sqrt{\mu})(x^2\sqrt{\lambda\mu}+y^2)+i(\sqrt{\lambda}-\sqrt{\mu})(z^2-w^2\sqrt{\mu\lambda})=0,\\ H_{\mathrm{II}}=-\sqrt{\mu}H_1+\sqrt{\lambda}H_{\mathrm{I}}\\ \qquad =(\sqrt{\lambda}-\sqrt{\mu})(-x^2\sqrt{\lambda\mu}+y^2)+i(\sqrt{\lambda}+\sqrt{\mu})(z^2+w^2\sqrt{\mu\lambda})=0.\end{cases}$$

From these equations we infer that the 24 chords of curvature become perfectly defined, if we choose a determinate value for the as yet arbitrary constant $\sqrt{\mu\lambda}$, but obviously, by doing so, we do not succeed in determining the biquadratic R, because this curve is represented by the two equations of H_1 and H_{I}, wherein

the constants λ and μ occur separately. Hence we have arrived at the theorem: There is an infinity of biquadratic curves that have the same chords of curvature in common with a given curve R.

Let us take $\sqrt{\mu\lambda} = \alpha^2$, then we are at once enabled to derive from the equations of H_1 and H_{I} the locus of these biquadratics. Any one of them is now represented by

$$\begin{cases} \lambda(x^2 - iw^2) + \quad (y^2 - iz^2) = 0, \\ \quad (x^2 + iw^2) - \lambda(y^2 + iz^2) = 0, \end{cases}$$

where λ is a variable parameter, and hence, by eliminating λ, we find the required locus to be the tetrahedral quartic surface

$$F = x^4 + y^4 + z^4 + w^4 = 0,$$

the equations of the 24 chords of curvature, right lines on F, being now in fact

I. (Chords, meeting O_2O_3, O_1O_4).

$$\begin{cases} x = \pm\,\alpha w, \\ y = \pm\,\alpha z, \end{cases} \quad \begin{cases} x = \pm\,\alpha^3 w, \\ y = \pm\,\alpha^3 z, \end{cases}$$

II. (Chords, meeting O_3O_1, O_2O_4).

$$\begin{cases} y = \pm\,\alpha w, \\ z = \pm\,\alpha x, \end{cases} \quad \begin{cases} y = \pm\,\alpha^3 w, \\ z = \pm\,\alpha^3 x, \end{cases}$$

III. (Chords, meeting O_1O_2, O_3O_4).

$$\begin{cases} z = \pm\,\alpha w, \\ x = \pm\,\alpha y, \end{cases} \quad \begin{cases} z = \pm\,\alpha^3 w, \\ x = \pm\,\alpha^3 y. \end{cases}$$

From the preceding it follows immediately that the surface F is a covariant of the curve R we originally started with, that is to say, the quaternary quartic F is a combinant of the two quadrics S and S' we have used to define analytically the curve R. And indeed, assuming S and S' respectively to be $H_1 + \mu H_{\mathrm{I}}$ and $H_1 + \mu' H_{\mathrm{I}}$, and denoting by U the reciprocal quadric of S with respect to S', and by U' the reciprocal quadric of S' with respect to S, there is no difficulty to show that F is identical with

$$2US' - 2U'S - \theta' S^2 - \theta S'^2,$$

a quantic which may be readily recognized as a combinant of S and S'.

Considering the two equations representing a biquadratic R on F, we see that two such curves cannot have a point in common, an arbitrary quadric however, described through R, intersects the surface F still in a second biquadratic R', who of course meets R in eight points. Hence there exists upon F a second

system of biquadratics R', such that a quadric containing a curve R also contains a curve R', and vice versa; or, what is the same thing, every chord of a curve R is at the same time a chord of one of the curves R'. This result can be established analytically. If the biquadratic R of the first system be given by the equations

$$\begin{cases} \lambda\,(x^2 - iw^2) + \quad (y^2 - iz^2) = 0\,, \\ \quad (x^2 + iw^2) - \lambda\,(y^2 + iz^2) = 0\,, \end{cases}$$

then we have only to alter the sign of iw^2 and to write μ for λ to obtain a second biquadratic R',

$$\begin{cases} \mu\,(x^2 + iw^2) + \quad (y^2 - iz^2) = 0\,, \\ \quad (x^2 - iw^2) - \mu\,(y^2 + iz^2) = 0\,, \end{cases}$$

also situated on the tetrahedral quartic surface, and there is no difficulty in seeing that through both curves we can describe the quadric

$$f = x^2(\lambda + \mu) + y^2(1 - \lambda\mu) - iz^2(1 + \mu\lambda) + iw^2(\mu - \lambda) = 0.$$

Now if we try to find the coordinates of the eight points P, in which the curves R and R' meet, we get the four equations

$$x^2 = \lambda + \mu, \quad y^2 = 1 - \lambda\mu, \quad z^2 = -i\,(1 + \mu\lambda), \quad w^2 = i\,(\mu - \lambda),$$

and so it becomes evident that the quadric f whereon both curves lie, is the polar quadric surface, with regard to F, of any one of these eight points P. Hence, since the tangents in P to the curve of intersection of F and this polar quadric are the two inflexional tangents in P, the two systems of biquadratics R and R' are proved to be identical with the asymptotic curves of the tetrahedral quartic surface F.

In drawing this conclusion we have arrived at a known result,* but at the same time the foregoing deductions allow us to add something to it, viz. the theorem that the tetrahedral quartic surface is completely determined as soon as one of its asymptotic curves is given.

And indeed if we start with the biquadratic R and describe through R an arbitrary quadric f, we can easily determine on f the biquadratic R' of the other system, it being sufficient to observe that in their meeting-points P, each of the curves R and R' touches one of the right lines of the quadric f.

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* This result was first obtained by Lie ("Ueber die Reciprocitäts-Verhältnisse des Reye'schen Complexes"; Göttinger Nachrichten, 1870, p. 53.)

Let us next consider the right line, who joins two points P and Q on the biquadratic R. As it meets the surface F in two points such that the polar quadric of either point passes through the other, we deduce from the ordinary theory of poles and polars that the line is cut harmonically in the two points P and Q and in the two remaining points where it meets the quartic surface again. All right lines, cut harmonically by the surface F, obviously constitute a complex of the sixth degree, and as the two cones, projecting from their common point P the two biquadratics R and R', make up together a cone of that degree, we may enunciate the theorem: The system of the chords of the asymptotic curves on the tetrahedral quartic surface is identical with the complex of lines, meeting the surface in two pairs of harmonically conjugate points.

Having thus noticed the most important properties of the systems of biquadratics on F, there remains to observe that, as well as the curves R, the curves R' have their chords of curvature in common. These 24 new right lines, completing the set of 48 right lines on F, will be found to be

I.
(Chords, meeting O_2O_3, O_1O_4).

$$\begin{cases} x = \pm \alpha w \\ y = \pm \alpha^3 z \end{cases}, \quad \begin{cases} x = \pm \alpha^3 w \\ y = \pm \alpha z \end{cases},$$

II.
(Chords, meeting O_3O_1, O_2O_4).

$$\begin{cases} y = \pm \alpha w \\ z = \pm \alpha^3 x \end{cases}, \quad \begin{cases} y = \pm \alpha^3 w \\ z = \pm \alpha x \end{cases},$$

III.
(Chords, meeting O_1O_2, O_3O_4).

$$\begin{cases} z = \pm \alpha w \\ x = \pm \alpha^3 y \end{cases}, \quad \begin{cases} z = \pm \alpha^3 w \\ x = \pm \alpha y \end{cases},$$

and from these equations we readily can get an idea of the position of these 24 lines with respect to those of the other system. So, for example, is it easily verified that the lines AF, FC, CH, HA and EB, BG, GD, DE, the sides of the two skew quadrilaterals $AFCH$ and $EBGD$, if drawn in the diagram above, would represent the eight chords of the first group. Moreover, it becomes now plain that we should consider the sides of any one of these skew quadrilaterals, formed by four chords of curvature, as a broken-up biquadratic R or R', each of the two systems R and R' containing six of these singular curves.

We conclude with some remarks about the invariants of the biquadratics on the surface F. A biquadratic is known to possess only one absolute invariant A, and if this invariant has the same value for two biquadratics, they can be transformed, one into another, by a linear substitution of the coordinates. In this

case we shall say that the two curves are of the same *type*, and we will now ask to find on F all the biquadratics of the same type as a given one R represented by the equations

$$\begin{cases} \lambda(x^2 - iw^2) + (y^2 - iz^2) = 0, \\ (x^2 + iw^2) - \lambda(y^2 + iz^2) = 0. \end{cases}$$

These two quadric surfaces determine a binary quartic

$$\Delta'\xi^4 + \theta'\xi^3 + \phi\xi^2 + \theta\xi + \Delta,$$

the invariants I and J of which we have to combine in order to find $I^3:16J^2$ as the expression of the absolute invariant A, belonging to the considered biquadratic. Working out the necessary calculations, the invariant A of the curve R with the parameter λ becomes

$$A = \frac{(\lambda^8 + 14\lambda^4 + 1)^3}{(\lambda^{12} - 33\lambda^8 - 33\lambda^4 + 1)^2}.$$

In this expression we recognize the well-known forms of the theory of the octahedron, and remembering the 24 so-called octahedral substitutions, we infer that we have a set of 24 curves R of the same type, the 24 corresponding values of the parameter λ being

$$e^{\frac{i\pi k}{2}}\lambda,\quad e^{\frac{i\pi k}{2}}\times\frac{1}{\lambda},\quad e^{\frac{i\pi k}{2}}\left(\frac{1+\lambda}{1-\lambda}\right),\quad e^{\frac{i\pi k}{2}}\left(\frac{1-\lambda}{1+\lambda}\right),\quad e^{\frac{i\pi k}{2}}\left(\frac{i+\lambda}{i-\lambda}\right),\quad e^{\frac{i\pi k}{2}}\left(\frac{i-\lambda}{i+\lambda}\right)$$
$$(k = 0,\ 1,\ 2,\ 3).$$

Quite the same result is arrived at when we consider the biquadratics R' of the second system

$$\begin{cases} \mu(x^2 + iw^2) + (y^2 - iz^2) = 0, \\ (x^2 - iw^2) - \mu(y^2 + iz^2) = 0. \end{cases}$$

Giving the parameter μ one of the 24 foregoing values, we obtain a further set of 24 curves R', all of the same type as the curve R with the parameter λ.

It thus being proved that there are in all 48 curves of a given type, we will now try to give some indications about their very symmetrical arrangement all over the surface F. In the first place we seek the locus of the meeting-points of a curve R and a curve R' of the same type, that is to say, we eliminate λ and μ between the equations

$$x^2 = \lambda + \mu,\quad y^2 = 1 - \lambda\mu,\quad z^2 = -i(1 + \mu\lambda),\quad w^2 = i(\mu - \lambda)$$

in the understanding that λ and μ are connected by one of the 24 octahedral

substitutions. Considering the several cases, one after another, we shall find that the required locus is made up by the 24 curves of intersection of the surface F with the following set of 24 quadrics Q, arranged in six groups of four:

$$\begin{cases} x^2 = 0, \\ y^2 = 0, \\ z^2 = 0, \\ w^2 = 0, \end{cases} \quad \begin{cases} x^2 + w^2 = 0, \\ x^2 - w^2 = 0, \\ y^2 + z^2 = 0, \\ y^2 - z^2 = 0, \end{cases} \quad \begin{cases} y^2 + w^2 = 0, \\ y^2 - w^2 = 0, \\ z^2 + x^2 = 0, \\ z^2 - x^2 = 0, \end{cases} \quad \begin{cases} z^2 + w^2 = 0, \\ z^2 - w^2 = 0, \\ x^2 - y^2 = 0, \\ x^2 + y^2 = 0, \end{cases}$$

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = 0, \\ x^2 - y^2 - z^2 + w^2 = 0, \\ -x^2 + y^2 - z^2 + w^2 = 0, \\ -x^2 - y^2 + z^2 + w^2 = 0, \end{cases} \quad \begin{cases} -x^2 + y^2 + z^2 + w^2 = 0, \\ x^2 - y^2 + z^2 + w^2 = 0, \\ x^2 + y^2 - z^2 + w^2 = 0, \\ x^2 + y^2 + z^2 - w^2 = 0. \end{cases}$$

Now we can, if we like, define these quadrics as geometrical loci. For it is possible to establish a correspondence between the six Vossian quadrics of a curve R and those belonging to a curve R' of the same type in such a manner that by moving the curve R along the tetrahedral quartic surface, the curve of intersection of each corresponding pair of Vossian quadrics describes one and the same quadric Q. There is however a more direct way to determine these quadrics Q if we only consider those points on the surface F where two right lines of different systems cross. Such points we have met already on the edges of the self-conjugate tetrahedron. In fact there are on each edge four points, where two chords of curvature of the biquadratics R cross two chords of curvature of the curves R'.

Counting these points, as is naturally, four times as a meeting-point of two right lines of different systems, and combining them with the 192 crossing-points found elsewhere on the surface, we obtain a total number of 288 of these special points. And by the help of them we can determine the quadrics Q, as any one of these surfaces passes through 48 of these points.

If, then, in this way these quadrics Q have been constructed, we can get by their aid some idea of the arrangement of the set of 48 biquadratics of a given type; for in fact these quadrics Q in some sort link together in their eight common points each pair of curves R and R'. And this arrangement becomes more simple and less intricate by considering the already indicated division of the quadrics Q into six groups of four. Starting with a given curve R and seeking for all biquadratics on F that can be linked to R, directly or indirectly,

by means of the four quadrics Q of one group, we shall come to the conclusion that no more than four curves of each system can be found. So then we get a perfectly regular configuration consisting of four curves R and four curves R', all of the same type, any pair of curves of different systems meeting each other on one of the four quadrics Q.

Again, taking in particular the first group of these quadrics Q, we see that, starting with any curve R, we can construct a special configuration in which each pair of consecutive curves is linked together by one of the faces of the self-conjugate tetrahedron, the two curves touching each other in four distinct points.

A final remark may be added about the linear substitutions of the coordinates by which the tetrahedral quartic surface F is transformed into itself. It is possible to show that the homographic transformation, which has the effect of changing a curve R with the parameter λ into one of the 47 other curves of the same type, does not involve the constant λ, and therefore by each of these transformations the surface F is transformed into itself. Now by introducing the elliptic argument, as we have done before, it is immediately apparent that a biquadratic can be transformed into itself in 32 different ways. Hence we are justified in stating that there are $32 \times 48 = 1536$ distinct linear substitutions of the coordinates which do not alter the equation of the surface F, a statement already made by Schur.*

* "Ueber Flächer vierter Ordnung," Math. Annalen, Bd. 20, p. 294.